A SHARP CONDITION FOR THE WELL-POSEDNESS OF THE LINEAR KDV-TYPE EQUATION

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arXiv:1209.1658v2 [math.AP] 19 Oct 2012

Abstract. An initial value problem for a very general linear equation of KdV-type is considered. Assuming non-degeneracy of the third derivative coefficient this problem is shown to be well-posed under a certain simple condition, which is an adaptation of Mizohata-type condition from the Schrödinger equation to the context of KdV. When this condition is violated ill-posedness is shown by an explicit construction. These results justify formal heuristics associated with dispersive problems and have applications to non-linear problems of KdV-type.

1. Introduction

This paper is concerned with the study of the equation

$$\begin{cases} \partial_t u + Lu = f \text{ for } (t,x) \in (0,T] \times \mathbb{R} \\ u(0,x) = u_0(x) \end{cases}, \text{ where } L = \sum_{i=0}^{3} a_i(t,x)\partial_x^i \tag{1}$$

where a_i are real-valued functions.
This is the most general linear form of the KdV, one of the most studied dispersive equations, and used as an important model in understanding behavior of linear and non-linear waves. Such an equation with non-constant dispersive coefficient a_3 describes nonisotropic dispersion and its study is of use for the quasi-linear analogues of (1).

Another motivation, for the study of the well-posedness of (1) is understanding the relative strength of dispersive and non-dispersive effects present in the equation. In particular, from the geometrical optics expansion for the equation, c.f. the classical book of Whitham [13], the dispersive coefficient a_3 guides the propagation of the wave packets, while the term $a_2\partial_x^2$ can lead to the growth of the amplitudes of the wave packets of (1). In light of these heuristics, it is natural to expect that well-posedness requires *non-degeneracy* of a_3, which prevents the collapse of the wave packets, namely $0 < \varepsilon \leq |a_3| \leq \frac{1}{\varepsilon}$ for some ε, and a condition on a_2 to ensure dispersion dominates anti-diffusion effects. Craig-Goodman [4] proved well-posedness in the Sobolev spaces H^s for $a_2 \equiv a_1 \equiv 0$ under the *non-degeneracy* of coefficient a_3 and ill-posedness for some degenerate cases of a_3. In a follow up paper, Craig-Kappeler-Strauss [3] proved well-posedness with non-degenerate dispersion and $-a_2 \geq 0$, as well as extensions to the quasi-linear analogues. These results were extended in [1] to allow for the "anti-diffusion" in a_2, as long as $\langle x\rangle^{\frac{1}{2}^+} |a_2| \leq C$, under some additional assumptions on other coefficients, and to systems of equations.

In the current paper, the condition on the diffusion coefficient a_2 is extended to a sharp one for the well-posedness in H^s, where well-posedness means *existence* of $C^0_{[0,T]}H^s$ distributional solutions

2010 *Mathematics Subject Classification.* Primary: 35Q53.
Key words and phrases. KdV, linear, dispersive, partial differential equations, energy method, Mizohata condition.

of (1), that are *unique* and *depend continuously on data* in the $C^0_{[0,T]}H^s$ topology. Namely a condition on the diffusion coefficient a_2 along the flow is obtained, that separates well-posedness from ill-posedness (in the sense of violating continuous dependence) of (1) with non-degenerate dispersion. This is qualitatively similar to the necessity of a Mizohato condition $|\sup_{x,t|\omega|=1}\int_0^t \Im b(x+s\omega)\cdot\omega ds| < \infty$ for the well-posedness Schrödinger equation $\partial_t u + i\triangle u + b(x)\nabla u = 0$ in [10], see also [5], [6], [9] and references therein for more refined results on the variable coefficient Schrödinger equation. The well-posedness is proved by the "gauged energy method" and the condition on the gauge captures the a_2 condition. Ill-posedness is proved by an explicit geometrical optics construction in the time independent coefficient case that has consequence for general coefficients.

While preparing this paper for publication, I have learned of a preprint by Ambrose-Wright [2] that treats an analogue of (1) in the periodic case. Their argument for the well-posedness is also based on the "gauged energy method", however in the case of $\mathbb{R}$ the smoothness of the coefficients does not imply integrability that is often needed. Additionally, this paper also proves that (1) possesses a local smoothing effect, which is not present in the periodic case. The ill-posedness result in [2] is done by a spectral method, using a change of variables that works in periodic case, but is unbounded on the real line.

The methods used in both positive and negative arguments of this paper can be extended to nonlinear problems, which will be a subject of future work.

The rest of the paper is organized as follows. In the section 2 the main results of the paper are stated. Well-posedness is proved in the section 3, and ill-posedness in section 4.

Some results of this paper were obtained during my Ph.D. studies at the University of Chicago, under the supervision of Carlos Kenig. I would like to thank Carlos Kenig and Cristian Rios for helpful discussions.

2. Main results.

The following functional space notation is used. Let
$\mathcal{B}^N_x = \{f(x) \in C^N(\mathbb{R}) : \partial^i_x f \in L^\infty \text{ for all } 0 \le i \le N\}$, $\mathcal{B} = \cap_n \mathcal{B}^N$, and
$H^s = \{f \in \mathscr{S}' : \|f\|_{H^s} = \|\langle\xi\rangle^s \hat{f}(\xi)\|_{L^2} < \infty\}$, where $\langle x\rangle = \sqrt{1+|x|^2}$.
Finally, when dealing with mixed norm spaces it is convenient to denote
$L^\infty_{[0,T]}X_x \equiv X_T$ for the spaces X as above.

The following assumptions are made for the coefficients of (1)

(A1): Dispersive coefficient $a_3(t,x)$ is non-degenerate. That is, there are constants $\Lambda \ge \lambda > 0$, such that

$\lambda \le |a_3(t,x)| \le \Lambda$

uniformly for $(x,t) \in \mathbb{R}\times[0,T]$.

(A2): Regularity of the coefficients. For all $N \ge 0$.

- $a_3 \in C^0_{[0,T]}\mathcal{B}^{N+3}_x \cap C^1_{[0,T]}\mathcal{B}^1_x$.
- $a_2 \in C^0_{[0,T]}\mathcal{B}^{N+2}_x \cap C^1_{[0,T]}\mathcal{B}^0_x$.
- $a_1 \in C^0_{[0,T]}\mathcal{B}^{N+1}_x$
- $a_0 \in C^0_{[0,T]}\mathcal{B}^N_x$.

(A3): Weak diffusion. $\int_0^x \frac{a_2(y,t)}{|a_3(y,t)|}dy \in C^1_{[0,T]}L^\infty_x$.

Note, that by $(A1)$ and $(A2)$, a_3 has a constant sign.

For $N \geq 0$ define

$$C_N = \|a_3\|_{L^\infty_T} + \|\frac{1}{a_3}\|_{L^\infty_T} + \sum_{i=0}^{3}\|a_i\|_{\mathcal{B}_T^{N+i}} + \sum_{i=2}^{3}\|\partial_t a_i\|_{L^\infty_T} + \|\int_0^x \frac{a_2(y,t)}{|a_3(y,t)|}dy\|_{L^\infty_T}$$
$$+ \|\partial_t \int_0^x \frac{a_2(y,t)}{|a_3(y,t)|}dy\|_{L^\infty_T}$$

For the well-posedness arguments, positive constants will depend on C_N for some N and will not be made explicit.

Theorem 1. *Suppose the coefficients of* (1) *satisfy* $(A1)$*-*$(A3)$*. Then for all* $s \in \mathbb{R}$*,* (1) *is well-posed in* H^s*. That is for any* $(u_0, f) \in H^s \times L^1_{[0,T]}H^s$ *there exists a unique* $u \in C^0_{[0,T]}H^s$ *satisfying* (1) *in the sense of distributions. In addition, there exists* $C = C(s)$

$$\sup_{0\leq t\leq T}\|u(t)\|_{H^s} \leq Ce^{CT}(\|u_0\|_{H^s} + \int_0^T \|f(t)\|_{H^s}dt) \tag{2}$$

Moreover, for any $\delta > \frac{1}{2}$*, the solution additionally satisfies* $u \in L^2_{[0,T]}H^{s+1}_{\langle x\rangle^{-2\delta}dx}$ *and there is a* $\tilde{C} = \tilde{C}(s,\delta)$

$$\|\langle x\rangle^{-\delta}\,\partial_x u\|_{L^2_{[0,T]}H^s_x} \leq \tilde{C}(1+\sqrt{T})e^{\tilde{C}T}(\|u_0\|_{H^s} + \int_0^T \|f(t)\|_{H^s}dt) \tag{3}$$

Estimate (2) implies continuous dependence for (1), while estimate (3) is a manifestation of a *local smoothing effect* of (1).

Remark 2. If in addition, $f \in C^0_{[0,T]}H^{s-3}$, then for $s > 3\frac{1}{2}$ the unique solution from the Theorem 1 is classical by the Sobolev embedding.

Remark 3. If the coefficients of (1), in addition, satisfy $(A1)$ - $(A3)$ on $[-T, 0]$, then the transformation of the equation by $t \to -t$ changes the sign of all a_i, while again preserving all of the assumptions. Therefore, Theorem 1 extends to $[-T, 0]$.
Moreover, the transformation $x \to -x$ in (1) changes the sign of a_i for odd i, but preserves the assumptions $(A1)$ - $(A3)$. Without of loss of generality $a_3 > 0$ will be assumed.

Ill-posedness result complements the Theorem 1 and is proved by a different method.

Theorem 4. *Suppose the coefficients of* (1) *satisfy* $(A1)$*,* $(A2)$ *and*

(A3N): $\sup_{x>0}\int_0^x \frac{a_2(y,0)}{|a_3(y,0)|}dy = \infty$

Then for all $T > 0$ *and* $s \in \mathbb{R}$ (1) *is ill-posed in* $C^0_{[0,T]}H^s$ *forward in time. More precisely, there is no continuous function* $C(t, t_0)$ *for* $0 \leq t_0 \leq t \leq T$*, such that*

$$\sup_{t_0\leq t\leq T}\|u(t)\|_{H^s} \leq C(t,t_0)\|u(t_0)\|_{H^s} \tag{4}$$

whenever u *solves* (1) *on* $[0, T]$ *with* $f \equiv 0$*. Equivalently* (2) *fails on any* $[0, T]$*.*

Remark 5. The transformation $x \to -x$ shows that $(A3N)$ is equivalent to

$$\sup_{x<0}\int_x^0 \frac{a_2(y,0)}{|a_3(y,0)|}dy = \infty.$$

However, the equivalence breaks down if absolute values are removed from a_3 in $(A3)$. Thus $a_3 > 0$ can be assumed without loss of generality, as long as $(A3N)$ is replaced with

(A3N'): $a_3 > 0$. Furthermore,

$$\sup_{x>0}\int_0^x \frac{a_2(y,0)}{a_3(y,0)}dy = \infty \text{ or } \sup_{x<0}\int_x^0 \frac{a_2(y,0)}{a_3(y,0)}dy = \infty$$

Remark 6. By reversing the time $t \to -t$ as in the Remark 3, Theorem 4 shows that

$$\sup_{x>0} \int_0^x \frac{a_2(y,0)}{|a_3(y,0)|} dy = -\infty$$

leads to ill-posedness on $[-T, 0]$. Thus the condition $\int_0^x \frac{a_2(y,0)}{|a_3(y,0)|} dy \in L^\infty$ is crucial for the well-posedness and the condition $(A3)$ for the Theorem 1 is sharp for well-posedness on $[-T, T]$.

3. Well-posedness

The main ingredient in the proof of the Theorem 1 is stated as the following Proposition, which is an *a priori* L^2 estimate for a slightly more general version of (1), that comes from commuting derivatives.

$$\begin{cases} \partial_t u + L_A u = f \text{ for } (t,x) \in (0,T] \times \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \quad \text{, where } L_A = L + A_0(t,x,\partial_x) \tag{5}$$

with L from (1). The following assumptions are made on $A_0 \in C^0_{[0,T]} S^0$, the Pseudo-Differential operator of standard symbol class of order 0 (Cf. Chapter VI of [11]):

- **(A4):** The S^0 semi-norms of A_0 are bounded for $t \in [0,T]$ and their size depends on constants C_N from $(A1)$–$(A3)$.

Proposition 7. *Suppose that the coefficients a_i of (1) satisfy $(A1)$–$(A3)$ and A_0 satisfies $(A4)$. Then there exists a constant C and for any $\delta > \frac{1}{2}$ there is a constant $\tilde{C}$, such that for any $u \in C^1_{[0,T]} L^2 \cap C^0_{[0,T]} H^3$, the triple (u, u_0, f) with u_0 and f defined by (5) satisfies*

$$\sup_{0 \le t \le T} \|u(t)\|_{L^2} \le C e^{CT} (\|u_0\|_{L^2} + \int_0^T \|f(t)\|_{L^2} dt) \tag{6}$$

$$\|\langle x \rangle^{-\delta} \partial_x u\|_{L^2_{[0,T]\times x}} \le \tilde{C}(1 + \sqrt{T}) e^{\tilde{C}T} (\|u_0\|_{L^2} + \int_0^T \|f(t)\|_{L^2} dt) \tag{7}$$

Remark 8. If $A_0 \equiv 0$, then $N = 0$ in $(A2)$ can be chosen for the Proposition 7.

The proof of the Proposition 7 is done by a change of variables (gauge) followed by the application of the energy estimates. The proof is broken into several preliminary results.

A gauge is a smooth invertible function, which for the purposes of the argument needs to have 3 bounded derivatives:

Definition 9. *A function $\phi \in C^0_{[0,T]} \mathcal{B}^3_x \cap C^1_{[0,T]} \mathcal{B}^0$ is called a* ***gauge****, if*

- *$\phi(x,t) > 0$ with $\frac{1}{\phi} \in L^\infty_{[0,T]\times\mathbb{R}}$.*
- *$\|\frac{1}{\phi}\|_{L^\infty_{[0,T]\times\mathbb{R}}} + \|\phi\|_{\mathcal{B}^3_T} + \|\partial_t \phi\|_{L^\infty_T} \le C(C_0, \delta)$ with C_N from $(A1)$–$(A3)$.*

Suppose that $\phi(x,t)$ is a *gauge*. Define

$$v = \phi^{-1} u$$

Definition 9 implies that $v \in C^1_{[0,T]} L^2 \cap C^0_{[0,T]} H^3$ if and only if $u \in C^1_{[0,T]} L^2 \cap C^0_{[0,T]} H^3$ and substitution of v into (5) gives:

$$\begin{cases} \partial_t v + L_\phi v = \phi^{-1} f \\ v(x,0) = \phi^{-1} u_0 \end{cases} \tag{8}$$

where

$$\begin{aligned} L_\phi = {} & a_3 \partial_x^3 + \left(a_2 + \phi^{-1} 3 a_3 \partial_x \phi\right) \partial_x^2 + \left(a_1 + \phi^{-1}(2 a_2 \partial_x \phi + 3 a_3 \partial_x^2 \phi)\right) \partial_x \\ & + \left(a_0 + \phi^{-1}(\partial_t \phi + a_1 \partial_x \phi + a_2 \partial_x^2 \phi + a_3 \partial_x^3 \phi)\right) I + \phi^{-1} A_0(\phi \_) \end{aligned}$$

Remark 10. From the definition of the gauge,

$$\|u\|_{L^2} \approx \|v\|_{L^2} \text{ and } \|\langle x\rangle^{-\delta}\partial_x u\|_{L^2_{[0,T]\times x}} \approx \sum_{i=0}^{1} \|\langle x\rangle^{-\delta}\partial_x^i v\|_{L^2_{[0,T]\times x}}$$

with comparability constants dependent only on the constant in the Definition 9. Therefore, to prove Proposition 7 it suffices to prove (6) and (7) for v satisfying (8).

The energy method involves multiplying (8) by v to estimate $\partial_t\|v\|^2_{L^2}$ by $\|v\|^2_{L^2}$:

$$\partial_t \int |v|^2 = -2Re(L_\phi v, v) + (f, \phi v)$$

The following Lemma summarizes the energy estimates for L or L_ϕ:

Lemma 11. *Consider an operator $L = a_3\partial_x^3 + a_2\partial_x^2 + a_1\partial_x + a_0$, where a_3–a_0 satisfy (A2). Then for $v \in C^0_{[0,T]}H^3$*

$$Re(Lv, v) = (\left[-a_2 + \frac{3}{2}\partial_x a_3\right]\partial_x v, \partial_x v) + (b_0 v, v)$$

for $b_0 = a_0 - \frac{1}{2}(\partial_x a_1 - \partial_x^2 a_2 + \partial_x^3 a_3)$, where (u, v) is an L^2_x pairing.

Proof of Lemma 11. The computation is immediate by computing the adjoint L^* of L using the Calculus of PDO. Alternatively, as L is a differential operator, the same computation can be also done by a repeated integration by parts. Indeed, the operator ∂_x^k is skew-adjoint for odd k, which implies that principal parts of odd order terms are eliminated by integration by parts. For example

$$(a_1\partial_x v, v) = -(v, a_1\,\partial_x v) - (\partial_x a_1\, v, v) = -\overline{(a_1\,\partial_x v, v)} - (\partial_x a_1\, v, v)$$

An identical computation shows

$$Re(a_3\partial_x^2 v, \partial_x v) = -\frac{1}{2}(\partial_x a_3\partial_x v, \partial_x v) \text{ and } Re(\partial_x^2 a_3\partial_x v, v) = -\frac{1}{2}(\partial_x^3\, a_3 v, v)$$

Using these identities and more integration by parts establishes

$$Re(a_3\partial_x^3 v, v) = \frac{3}{2}(\partial_x a_3\,\partial_x v, \partial_x v) - \frac{1}{2}(\partial_x^3\, a_3 v, v)$$

A similar analysis for $Re(a_2\,\partial_x^2 v, v)$ completes the proof. ☐

Applying Lemma 11 to L_ϕ, shows that the only term of order higher than 0 is $\left([2a_2 + \frac{6a_3\partial_x\phi}{\phi} - 3\partial_x a_3]\partial_x v, \partial_x v\right)$. Thus, if this term were negative, an a priori estimate would be obtained for v. This motivates the choice of a gauge ϕ that should satisfy

$$2a_2 + \phi^{-1}6a_3\partial_x\phi - 3\partial_x a_3 \leq 0$$

A choice of equality in this equation can be made and this choice is enough for the estimate (6), but by exploiting the inequality the local smoothing estimate (7) is proved. The exact choice of a gauge is summarized in the following Lemma

Lemma 12. *For $\delta > \frac{1}{2}$, let $\phi(x,t)$ be a solution of the ODE*

$$\begin{cases} 6a\partial_x\phi = \left(3\partial_x a - c_\delta\langle x\rangle^{-2\delta} - 2a_2\right)\phi \\ \phi(t, 0) = 1 \end{cases}$$

where $c_\delta = 0$ or 1. Then ϕ is a gauge in the sense of the Definition 9, and is independent of δ if $c_\delta = 0$.

Proof. The ODE for ϕ is solved explicitly as

$$\phi(x,t) = \sqrt{\frac{a(x,t)}{a(t,0)}} e^{-\int_0^x \frac{a_2(y,t)}{3a_3(y,t)}dy} e^{-\int_0^x \frac{c_\delta dy}{6a_3(y,t)\langle y\rangle^{2\delta}}}$$

By $(A3)$ $e^{-\int_0^x \frac{a_2(y,t)}{3a_3(y,t)}dy} \approx 1$. $(A1)$ implies $\sqrt{\frac{a(x,t)}{a(t,0)}} \approx 1$.
Finally, as $\delta > \frac{1}{2}$,

$$e^{-\int_0^x \frac{c_\delta dy}{6a_3(y,t)\langle y\rangle^{2\delta}}} = \begin{cases} 1, & \text{if } c_\delta = 0 \\ C(\delta), & \text{if } c_\delta = 1 \end{cases}$$

A computation for $\partial_t \phi$ and $\partial_x^j \phi$ for $j = 1$, 2 and 3 and using $(A1)$–$(A3)$ finishes the proof. □

Proof of Proposition 7. By the Remark 10 it suffices to prove the Proposition for v satisfying (8).

Applying the Lemma 11 for L_ϕ implies that

$$\partial_t \int |v|^2 dx = (\left[2a_2 + \frac{6a_3\partial_x\phi}{\phi} - 3\partial_x a_3\right]\partial_x v, \partial_x v) + (\tilde{b}_0 v, v)$$
$$- 2Re(A_0(\phi v), \phi v) + (f, \phi v)$$

where $\tilde{b}_0$ is obtained from the Lemma 11 applied to L_ϕ. With ϕ chosen from the Lemma 12, this implies

$$\partial_t \int |v|^2 dx \le -c_\delta(\langle x\rangle^{-2\delta} v, v) + (\tilde{b}_0 v, v) - 2Re(A_0(\phi v), \phi v) + (f, \phi v)$$

By (A4), $A_0 : L^2 \to L^2$ is bounded. Moreover, by the Definition 9 and $(A2)$, $\phi \in L^\infty$ and $\tilde{b}_0 \in L^\infty$. Hence

$$\partial_t \int |v|^2 \le C(\int |v|^2 dx + \|v\|_{L^2}\|f\|_{L^2}) - \|\langle x\rangle^{-\delta} \partial_x v\|_{L^2}^2$$

For $c_\delta = 0$ an application of Grownwall Lemma implies (6) for v.

Whereas moving $\partial_x v$ term to the left hand side for $c_\delta = 1$ and integrating in time gives

$$\int_0^T \|\langle x\rangle^{-\delta} \partial_x v\|^2 dt \le C\int_0^T (\int |v|^2 dx + \|v\|_{L^2}\|f\|_{L^2})dt + \|v_0\|_{L^2}^2 - \|v\|_{L^2}^2$$
$$\le (C(1+T) - 1) \sup_{0\le t\le T} \|v(t)\|_{L^2}^2 + \|v_0\|_{L^2}^2 + (\int_0^T \|f(t)\|_{L^2} dt)^2$$

Using (6) completes the proof of (7). □

Proposition 7 can be strengthened to an H^s estimate.

Proposition 13. *Let L be as in* (1), *whose coefficients a_i satisfy* $(A1)$–$(A3)$. *Then for any $s \in \mathbb{R}$ there exist constants $C(s)$ and $\tilde{C}(s,\delta)$ for any $\delta > \frac{1}{2}$, such that for any $u \in C^1_{[0,T]}H^s \cap C^0_{[0,T]}H^{s+3}$ the following estimates hold*

$$\sup_{0\le t\le T} \|u(t)\|_{H^s_x} \le Ce^{CT}(\|u(0)\|_{H^s_x} + \int_0^T \|\partial_t u + Lu\|_{H^s_x} dt) \tag{9}$$

$$\sup_{0\le t\le T} \|u(t)\|_{H^s_x} \le Ce^{CT}(\|u(T)\|_{H^s_x} + \int_0^T \|-\partial_t u + L^* u\|_{H^s_x} dt)$$

where L^* is the adjoint of L. Moreover

$$\|\langle x\rangle^{-\delta}\partial_x u\|_{L^2_{[0,T]}H^s_x} \le \tilde{C}(1+\sqrt{T})e^{\tilde{C}T}(\|u_0\|_{H^s} + \int_0^T \|f(t)\|_{H^s}dt)$$

Corollary 14. *By the Theorem 23.1.2 on page 387 in* [8], *the proof of the Theorem 1 reduces to the Proportion 13.*

The Proposition 13 is reduced to the Proposition 7. Observe, that

$$f = \partial_t u + Lu \text{ if and only if } J^s f = \partial_t J^s u + LJ^s u + [J^s L]J^{-s}J^s u$$

where J^s is a Pseudo Differential Operator with symbol $\langle\xi\rangle^s$. Therefore to prove (9) it suffices to show that the Proposition 7 applies to the operator $\tilde{L} = L + [J^s L]J^{-s}$.

Lemma 15. *Let $\tilde{L} = L + [J^s L]J^{-s}$ with L from* (1) *that satisfies* $(A1)$ *and* $(A2)$. *Then*

$$\tilde{L} = a_3\partial_x^3 + a_0 + \sum_{i=1}^{2}(a_i + \tilde{a}_i)\partial_x^i + A_s(t,x,\partial_x)$$
$$\text{with } \tilde{a}_2 = s\partial_x a_3 \text{ and } \tilde{a}_1 = s\partial_x a_2 + \frac{s(s-1)}{2}\partial_x^2 a_3 \tag{10}$$

where $A_s \in S^0$, whose semi-norms depend on the coefficient bounds $(A2)$ *for $N = N(s)$ and hence satisfies* $(A4)$.

Furthermore, the coefficients $\tilde{a}_i$ for $i = 1$, 2 satisfy $(A2)$*–*$(A3)$.

Proof. From the first term in the Calculus of PDO $[J^s L]J^{-s} \in S^2$. A further expansion of $[J^s, a_3\partial_x^3]$ gives:

$$\sigma([J^s, a_3\partial_x^3]) = \sum_{1\le|\alpha|\le 2} \frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha \langle\xi\rangle^s \partial_x^\alpha(a_3(i\xi)^3) \mod S^s$$
$$= s\partial_x a_3(i\xi)^2\langle\xi\rangle^s + \frac{s(s-1)}{2}\partial_x^2 a_3 i\xi\langle\xi\rangle^s \mod S^s$$

where the substitution $\xi^2 = \langle\xi\rangle^2 - 1$ was used and the terms of order s were absorbed into the remainder. Performing a similar computation for the remaining terms in $[J^s L]$ and composition with J^{-s} verifies (10).

It is immediate from (10) that $\tilde{a}_i$ satisfies $(A2)$. To verify $(A3)$ observe that

$$\int_0^x \frac{\tilde{a}_2(y,t)}{|a_3(y,t)|}dy = s\,\mathrm{sign}(a_3)\log\frac{a_3(x,t)}{a_3(0,t)} \in C^1_{[0,T]}L^\infty_x$$

by $(A1)$ and $(A2)$. □

Remark 16. A simple computation shows that the adjoint L^* of the operator L from (1) is

$$L^* = -a_3\partial_x^3 + (a_2 - 3\partial_x a_3)\partial_x^2 + (a_1 + 2\partial_x a_2 - 3\partial_x^2 a_3)\partial_x$$
$$+ (a_0 - \partial_x a_1 + \partial_x^2 a_2 - \partial_x^3 a_3)$$

whereas a substitution $t \to T - t$ transforms (1) to

$$\begin{cases} -\partial_t u(T-t) + Lu(T-t) = f(T-t) \\ u(T-t)\mid_{t=0} = u(T) \end{cases}$$

Both L^* and $L(T-t)$ satisfy $(A1)$–$(A3)$.

Corollary 17. *Lemma 15, Remark 16 and the Proposition 7 imply the Proposition 13.*

This completes the proof of Theorem 1 by the Corollary 14.

4. Ill-posedness

Ill-posedness is proved by justifying the formal geometrical optics argument for a special choice of initial data. The case of time independent dispersive coefficient is commonly studied in the literature (Cf. [5] and [6] for the variable dispersion Schrödinger equation). The same simplification is done in this work and the construction is flexible enough to be useful in the generality of (1). Thus (1) is replaced by

$$\begin{cases} \partial_t u + L_0 u = f \text{ for } (t,x) \in (0,T] \times \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \tag{11}$$

for $L_0 = \sum_{i=2}^{3} a_i(x,0)\partial_x^i + \sum_{i=0}^{1} a_i(t,x)\partial_x^i$ with coefficient a_i from L. The coefficients a_1 and a_0 are insignificant for the argument and no change is done for them. To simplify notation, denote $a_i(x) = a_i(x,0)$ for $i = 2$ and 3.
The geometrical optics argument involves making an ansatz for the solution of (1) of the form $u = e^{iS}w$ or (11) and converting them into a system of simpler equations. Indeed, a substitution of such ansatz into (11) gives

$$\begin{aligned} f &= e^{iS}(i\partial_t S - ia_3\partial_x S^3) + e^{iS}\left\{\partial_t w - 3a_3\partial_x S^2\partial_x w - [3a_3\partial_x S\partial_x^2 S + a_2\partial_x S^2]w\right\} \\ &+ e^{iS}\left\{a_3(i\partial_x^3 S\cdot w + 3i\partial_x^2 S\partial_x w + 3i\partial_x S\partial_x^2 w + \partial_x^3 w) + a_2(i\partial_x^2 S\cdot w + 2i\partial_x S\partial_x w)\right\} \\ &+ e^{iS}\left\{a_2\partial_x^2 w + a_1(i\partial_x S\cdot w + \partial_x w) + a_0 w\right\} \end{aligned}$$

Formally assuming that $\partial_x S \approx \xi$ and $w \approx 1$ for $\xi \gg 1$, leads to an eikonal equation for S to eliminate ξ^3 terms, a transport equation for w to eliminate ξ^2 terms and the remaining terms left in f. More precisely, an ansatz of the form $u = e^{iS}w$ solves (11) for appropriate u_0 and f provided

$$\begin{cases} \partial_t S - a_3(x)(\partial_x S)^3 = 0 \\ S(\xi,0,x) = \xi\int_0^x a_3^{-\frac{1}{3}}(y)dy \end{cases} \tag{12}$$

$$\begin{cases} \partial_t w - 3a_3(x)(\partial_x S)^2\partial_x w - [3a_3(x)\partial_x S\partial_x^2 S + a_2(x)(\partial_x S)^2]w = 0 \\ w(\xi,0,x) = w_0(x) \end{cases} \tag{13}$$

$$\begin{aligned} f &= e^{iS}a_3\left[i\partial_x^3 S\cdot w + 3i\partial_x^2 S\partial_x w + 3i\partial_x S\partial_x^2 w + \partial_x^3 w\right] \\ &+ e^{iS}\left[a_2(i\partial_x^2 S\cdot w + 2i\partial_x S\partial_x w + \partial_x^2 w) + a_1(i\partial_x S\cdot w + \partial_x w) + a_0 w\right] \end{aligned} \tag{14}$$

The first two equations are solved by the method of characteristics, Cf. [7], and the third is completely determined by S and w. However, the method of characteristics, in general, gives only local solutions, and it was important to make a choice for $S(\xi,0,x)$ in (12) to ensure S has a solution on $[0,\infty)\times\mathbb{R}$.

When dispersion coefficient is constant, which after rescaling is equivalent to $a_3(x,t) \equiv 1$, (12) is solved with a simple use of the dispersive relation by setting $S = \xi^3 t + \xi x$. Using an analogue of $(A3N)$ in this case, ill-posedness was proved in [1], by analogy with the Schrödinger equation as presented in lecture notes by Carlos Kenig on "The Cauchy Problem for the Quasi-linear Schrödinger Equation". This analysis is expanded below to account for non-constant dispersion.

The main ingredient in the proof of the Theorem 4 is the following Theorem, whose proof is based on the geometrical optics ansatz above.

Theorem 18. *Suppose the coefficients of* (11) *satisfy* $(A1)$, $(A2)$ *and* $(A3N)$ *as given in the Theorem 4.*

Let $n \in \mathbb{N}$. Then there exists a $T_n > 0$ and $u = u_n(\xi, t, x) \in C_\xi^N C^1_{[0,T]} C_0^{N+3}(\mathbb{R})$, such that

$$\|u_n(\xi, \frac{T_n}{\xi^2}, x)\|_{L^2_x} \geq 2n \text{ and } \|u(\xi, 0, x)\|_{L^2_x} = 1$$

$$\sup_{0 \leq t \leq \frac{T_n}{\xi^2}} \|[\partial_t u_n + L_0 u_n](\xi, t, x)\|_{L^2_x} \leq C_n \xi \tag{15}$$

$$\sup_{0 \leq t \leq \frac{T_n}{\xi^2}} \|u_n(\xi, t, x)\|_{H^i_x} \leq C_n \xi^i \text{ for } i = 0, \ldots, 3.$$

Moreover, adding a Pseudo-differential operator $A_0 \in S^0$, that satisfies $(A4)$, to L_0 does not change the conclusion above.

Theorem 18 is proved after a number of preparatory Lemmas in the subsection. After that the Theorem 4 is proved in the section 4.2.

4.1. **Analysis of** (12)–(14)**.** Note, that the choice of $S(\xi, 0, x) = \xi \int_0^x a_3^{-\frac{1}{3}}(y) dy$ is well-defined by $(A1)$, but will not be bounded as $x \to \infty$. This choice makes $\partial_t S(x, 0, \xi) = \xi^3$ constant for all x from the equation (12). If $S(x,t)$ solves (12), then $\Xi(t) = \partial_x S(x(t), t)$, $\omega(t) = \partial_t S(x(t), t)$ and $x(t)$ satisfy

$$H(\omega, x, \Xi) = \omega - a_3(x)\Xi^3 = 0$$

To achieve $\frac{dH}{dt} = 0$ a following system of "characteristic" ODEs is selected, which is later used to construct S:

$$\begin{cases} \frac{dx}{dt} = \frac{\partial H}{\partial \Xi} = -3a_3(x)\Xi^2 \\ x(0) = x_0 \end{cases} \qquad \begin{cases} \frac{d\Xi}{dt} = -\frac{\partial H}{\partial x} = \partial_x a_3(x)\Xi^3 \\ \Xi(0) = \xi_0 = \xi a_3^{-\frac{1}{3}}(x_0) \end{cases}$$
$$\begin{cases} \frac{d\omega}{dt} = -\frac{\partial H}{\partial t} = 0 \\ \omega(0) = a_3(x_0)\xi_0^3 = \xi^3 \end{cases} \tag{16}$$

The last two equations in (16) are easy to solve explicitly: $\omega(t) = \xi^3$ and $\Xi(t) = \xi a_3^{-\frac{1}{3}}(x(t))$. This simplifies the system above to a single ODE, where an index ξ is added to $x(t) = x_\xi(t)$ to emphasize the ξ dependence

$$\begin{cases} \frac{dx_\xi}{dt} = -3\xi^2 a_3^{\frac{1}{3}}(x_\xi) \\ x_\xi(0) = x_0 \end{cases} \tag{17}$$

$\xi \neq 0$ for the construction, so x_0 is unambiguous. As the bounds on $a_3^{\frac{1}{3}}(x)$ and $\partial_x[a_3^{\frac{1}{3}}(x)]$ are uniform in x by $(A1)$ and $(A2)$, Picard iteration gives a unique global solution for each x_0, Cf. Chapter 1 of [12].

Remark 19. Observe, that re-scaling time $t \to \frac{t}{\xi^2}$ for $\xi \neq 0$ implies by uniqueness of solutions to (17), that

$$x_\xi(\frac{t}{\xi^2}) = x_1(t)$$

where $x_1(t)$ satisfies (17) for $\xi = 1$. Using this rescaling, the index $\xi \neq 0$ can be suppressed to $x(t) = x_1(t)$

The properties of this ODE are summarized as follows, where as in the Remark 5, $a_3 > 0$ is assumed.

Lemma 20. *There exists a global in time flow $x_0 \to x(t)$ that depends smoothly on data with*

$$\frac{\partial x(t)}{\partial x_0} = \sqrt[3]{\frac{a_3(x(t))}{a_3(x_0)}} \tag{18}$$

As (17) *is autonomous in time, for each* (x, T) *there exists a unique* x_0*, such that* $x(t)$ *satisfies* (17) *and* $x_1(T) = x$.
Moreover, the flow is geodesic on $\mathbb{R}$*, that is for every* x_0*,* x *there exists a unique* T*, such that* $x(t)$ *satisfies* (17) *for all* t *and* $x(T) = x$ *and*

$$x(T) - x_0 \approx -T \tag{19}$$

with proportionality constants dependent on the $(A1)$ *bounds.*

Proof. The global existence follows from Picard iteration as observed before the Remark 19. Reversing the time direction $t \to T - t$ in (17) shows the unique dependence of x_0 on (x, T) from the equation.

By the Implicit Function Theorem for a fixed $x = x(x_0(x, t), t)$:

$$\frac{\partial x}{\partial x_0}\frac{dx_0(x,t)}{dt} + \frac{dx}{dt} = 0$$

which gives (18) using (17).

Finally, integrating (17) in time, using $(A1)$ and continuity of $x_1(t)$ proves the *geodesic* property. □

Remark 21. Using Remark 19, Lemma 20 holds for the unrescaled flow $x_\xi(t) = x_1(\xi^2 t)$ with obvious changes. That is given (ξ, x, t) there is a unique x_0, such that the solution of (17) satisfies $x_\xi(t) = x$, (18) holds and the flow map in (17) is geodesic with (19) replaced by $x_\xi(T, x_0) - x_0 \approx -T\xi^2$.

This leads to the following construction, which reconstructs S from the characteristic equations (16)

Lemma 22. *Define* $S(\xi, x, t) = S(\xi, x_0, 0) + \int_0^t \Xi(s)\frac{dx_\xi}{dt}(s) + \omega(s)ds$*, where* x_0 *is defined as in the Remark 21 and* $x_\xi(s)$*,* $\Xi(s)$ *and* $\omega(s)$ *solve* (16)*. Then* S *is well-defined on* $\mathbb{R} \times \mathbb{R}$*, satisfies* $\frac{\partial S}{\partial x}(x, t) = \Xi(t)$*,* $\frac{\partial S}{\partial t}(x, t) = \omega$ *and solves* (12) *for all* (x, t)*.*

Proof. Using (16) and the choice of data $S(\xi, x, 0)$, $S(\xi, x, t)$ can be simplified to

$$S(\xi, x, t) = \xi\int_0^{x_0} a_3^{-\frac{1}{3}}(y)dy - 2\xi^3 t$$

Differentiating this formula using $\frac{\partial x_0}{\partial t}$ and $\frac{\partial x_0}{\partial x}$ using the Lemma 20 and Remark 21 completes the proof. □

Next (13) is analyzed. Let $x(t) = x_\xi(t)$ and $\Xi(t)$ be as in (16). With this notation (13) becomes

$$\begin{cases} \partial_t w(\xi, t, x(t)) - 3a_3(x(t))\Xi^2(t)\partial_x w(t, x(t)) \\ = [3a_3(x(t))\Xi(t)\partial_x\Xi(t) + a_2(x(t))\Xi^2(t)]w(t, x(t)) \\ w(\xi, 0, x_0) = w_0(x_0) \end{cases}$$

which is an ODE in t as $\frac{dx_\xi}{dt} = -3a_3(x(t))\Xi^2(t)$. Solving this ODE for $w(t, x(t))$ gives

$$w(\xi, t, x) = \sqrt[3]{\frac{a_3(x)}{a_3(x_0)}}e^{\frac{1}{3}\int_x^{x_0}\frac{a_2(y)}{a_3(y)}dy}\, w_0(x_0) \tag{20}$$

where the identity $\partial_x\Xi(t) = \frac{-\partial_x a_3(x)}{3a_3(x)}\Xi(t)$, derived from $\Xi(t) = \xi a_3^{-\frac{1}{3}}(x(t))$, was used and where x_0 is defined as in the Remark 21.

Lemma 23. *Assume (A1), (A2) and (A3N'). Let $n \in \mathbb{N}$. Then there exists a $T_n > 0$ and $w_0^n(x) \in C_0^\infty(\mathbb{R})$ with $\|w_0^n(x)\|_{L_x^2} = 1$, such that the solution w^n from (20) corresponding to w_0^n satisfies*

$$\|w_n(\xi, \frac{T_n}{\xi^2}, x)\|_{L_x^2} \geq 2n \text{ and } \sup_{0 \leq t \leq \frac{T_n}{\xi^2}} \|w_n(\xi, t, x)\|_{H^i} \leq C_n(i),\ i \in \mathbb{N}$$

Proof. Rescaling time as in the Remark 21 it suffices to consider the case of $\xi = 1$. By (A3N') there exists $x_1^n < x_0^n$, such that

$$\sqrt{\frac{a_3(x)}{a_3(x_0)}} e^{\frac{1}{3}\int_{x_1^n}^{x_0^n} \frac{a_2(y)}{a_3(y)} dy} \geq 3n$$

By the Lemma 20, there exists a $T_n > 0$ such that $x_1(t)$ satisfies (17) with x_0^n for data and $x_1(T_n) = x_1^n$. Furthermore, by Lemma 20 and $(A2)$ there exists an interval I_n centered at x_0^n, such that for $x_0 \in I_n$

$$\sqrt{\frac{a_3(x)}{a_3(x_0)}} e^{\frac{1}{3}\int_{x_1(T_n, x_0)}^{x_0} \frac{a_2(y)}{a_3(y)} dy} \geq 2n \tag{21}$$

This condition mirrors the condition for ϕ from Lemma 12 failing to be a gauge.
Let $w_0^n \in C_0^\infty(I_n)$ with $\|w_0^n(x)\|_{L_x^2} = 1$. In particular, there exists a $C_n(s)$, such that $\|w_0^n(x)\|_{H^s} \leq C_n(s)$ for $s \geq 0$.
Define

$$I_n^t = \{x(t) : x(t) \text{ satisfies (17) for } x_0 \in I_n\}$$

Then from (20) and Lemma 20 $w(1, t\_)$ is compactly supported in I_n^t for all $t \geq 0$ and in particular, (21) is satisfied for $x \in \operatorname{supp} w(1, t, \_)$. Therefore,

$$\|w_n(1, T_n, x)\|_{L_x^2}^2 = \int_{I_n^t} \left(\frac{a_3(x)}{a_3(x_0)}\right)^{\frac{2}{3}} e^{\frac{2}{3}\int_x^{x_0(x, T_n)} \frac{a_2(y)}{a_3(y)} dy} |w_0(x_0)|^2 dx$$

Change of variables from x to x_0 using (18) to get

$$\|w_n(1, T_n, x)\|_{L_x^2}^2 = \int_{I_n} \frac{a_3(x)}{a_3(x_0)} e^{\frac{2}{3}\int_{x(x_0, T_n)}^{x_0} \frac{a_2(y)}{a_3(y)} dy} |w_0(x_0)|^2 dx_0 \geq 4n^2 \|w_0\|_{L_x^2}^2 \tag{22}$$

Where the last line follows by (21).

Thus it suffices to estimate $\|w_n(1, t, x)\|_{H^3}$ for $0 \leq t \leq T$ to finish the proof. By (19), (A1) and (A2) $\int_{x(x_0, t)}^{x_0} \frac{a_2(y)}{a_3(y)} dy \leq Ct$. Exploiting this fact in the equality part of (22) for some $0 \leq t \leq T_n$ instead of T_n implies that

$$\|w_n(1, t, x)\|_{L_x^2}^2 \leq C \int_{I_n} e^{Ct} |w_0(x_0)|^2 dx_0 \leq C(t)$$

$\partial_x^i w_n$ is estimated similarly after differentiation of (20) in x using (18) to handle derivatives of terms with x_0 . □

Proof of Theorem 18. By equations (12)–(14) $u_n(\xi, \_) = e^{iS(\xi, \_)} w_n(\xi, \_)$ is a solution of (11). S is real, so by Lemma 23

$$\|u_n(\xi, \frac{T_n}{\xi^2}, x)\|_{L_x^2} = \|w_n(\xi, \frac{T_n}{\xi^2}, x)\|_{L_x^2} \geq 2n$$

and $\|u_n(\xi, 0)\|_{L_x^2} = 1$. Therefore to finish the proof it suffices to show

$$\sup_{0 \leq t \leq \frac{T}{\xi^2}} \|f(\xi, t, x)\|_{L_x^2} \leq C_n \xi \text{ and } \sup_{0 \leq t \leq \frac{T_n}{\xi^2}} \|u_n(\xi, t, x)\|_{H_x^i} \leq C_n(i) \xi^i,\ i \in \mathbb{N}$$

for f from (14). From Cauchy-Schwarz

$$\|f(\xi,t,x)\|_{L^2_x} \le C\|\partial_x S(\xi,t,x)\|_{W^{2,\infty}_x}\|w_n\|_{H^3}$$

By the Lemma 22, $\partial_x S(\xi,t,x) = \xi a_3^{-\frac{1}{3}}(x)$ and hence $\|\partial_x S(\xi,t,x)\|_{W^{2,\infty}_x} \le C\xi$. Meanwhile, by Lemma 23 $\sup_{0\le t\le \frac{T_n}{\xi^2}}\|w_n\|_{H^i} \le C_n(i)$. Note, that adding a Pseudo-Differential operator $A_0 \in S^0$ replaces the coefficient a_0 in (14) by A_0 and the analysis above applies.

Similarly, differentiate $u = e^{iS}w$ and use the estimates of S and w to obtain

$$\sup_{0\le t\le \frac{T_n}{\xi^2}}\|u_n(\xi,t,x)\|_{H^i_x} \le C\|\partial_x S(\xi,t,x)\|^i_{W^{i,\infty}_x}\sup_{0\le t\le \frac{T_n}{\xi^2}}\|w_n\|_{H^i} \le C_n(i)\xi^i \quad \square$$

4.2. **Proof of illposedness.**

Lemma 24. *Suppose (A1), (A2) and (A3N) hold. Let $s\in\mathbb{R}$ and $n\in\mathbb{N}$. Then there exist a sequence $T'_n\to 0$, and a sequence of functions v^s_n, such that*

$$\sup_{0\le t\le T'_n}\|v^s_n(t)\|_{H^s} \ge n(\|v^s_n(0)\|_{H^s_x} + \int_0^{T'_n}\|\partial_t v^s_n + Lv^s_n\|_{H^s}dt) \tag{23}$$

Corollary 25. *Lemma 24 implies that (2) fails or, more generally, for any $T>0$ there is no non-decreasing function $C(T')$ on $[0,T]$, such that*

$$\sup_{0\le t\le T'}\|u(t)\|_{H^s} \le C(T')(\|u_0\|_{H^s} + \int_0^{T'}\|f(t)\|_{H^s}dt) \tag{24}$$

holds for all $u\in C^0_{[0,T]}H^s$ solving (1).

Proof. Assuming (24), for the sake of contradiction, and using (23), implies that $C(0)\ge C(T_n)\ge n$ for all $n\in\mathbb{N}$, which is impossible. $\square$

Proof of Lemma 24. Consider, first the case of $s=0$. The solution of (11) u_n from Theorem 18 solves

$$\begin{cases}\partial_t u_n + Lu_n = f_n + g_n\\ u_n(0) = e^{iS}w^n_0(x)\end{cases}$$

where $f_n := \partial_t u_n + L_0 u_n$ and $g_n = (L-L_0)u_n$, where L and L_0 are from (1) and (11), respectively. From (11) and the Fundamental Theorem of Calculus

$$(L_0-L)u_n(t) = \sum_{i=2}^3(a_i(x,t)-a_i(x,0))\partial^i_x u(t) = t\sum_{i=2}^3\int_0^1\partial_t a_i(x,st)\cdot ds\partial^i_x u(t)$$

and hence by $(A2)$ and the Theorem 18 for $0\le t\le fracT_n\xi^2$

$$\|g_n(t)\|_{L^2} \le Ct\|u_n(t)\|_{H^3} \le C_n t\xi^3$$

Combining this estimate with $\|f_n\|$ from (15) and integrating in time gives

$$\int_0^{\frac{T_n}{\xi^2}}\|f_n+g_n\|dt \le C_n(\frac{T_n}{\xi}+\frac{T_n^2}{\xi})$$

By the Theorem 18, $\|u_n(\xi,\frac{T_n}{\xi^2})\|_{L^2_x}\ge 2n$. Let $\xi=\xi_0$ be large enough, so that $T'_n := \frac{T_n}{\xi_0^2}\le\frac{1}{n}$ and $\int_0^{T'_n}\|f_n+g_n\|dt\le 1$. Define $v^0_n(x,t) = u_n(\xi_0,x,t)$ proves (23) for $s=0$.

For the general $s\in\mathbb{R}$, commute J^s with L as in Lemma 15: $J^s(\partial_t+L) = (\partial_t+\tilde{L})J^s$, where $\tilde{L} = L+[J^sL]J^{-s}$. Lemma 15 implies that (A1), (A2) and (A3N) are valid for $\tilde{L}$. Define $v^s_n =$

$J^{-s}u_n(\xi_0, x, t)$ where u_n is a function from Theorem 4 with L replaced by $\tilde{L}$. This reduces the proof of (23) for arbitrary s to $s = 0$ as $\|v_n^s(t)\|_{H^s} = \|u_n(\xi_0, t)\|_{L^2}$ and $\|f\|_{H^s} = \|(\partial_t + \tilde{L})u_n(\xi_0, t)\|_{L^2}$. $\square$

Corollary 26. *Assuming (A3N) as above implies, that* (1) *is ill-posed in* H^s *the sense of Theorem 4.*

Proof. Suppose, for the sake of contradiction, that (4) holds for some $[0, T]$ and some continuous function $C(t_0, t)$ for $0 \le t_0 \le t \le T$. Define a non-decreasing function $C(T') = \sup_{0 \le t_0 \le t \le T'} C(t_0, t)$. Then by the Duhamel principle every solution of (1) satisfies

$$u(t) = S(t, 0)u_0 + \int_0^t S(t, t_0) f(t_0) dt_0$$

where $u(t) = S(t, t_0)g$ solves (1) on $[t_0, T]$ with data $u(t_0) = g$ and $f \equiv 0$. Moreover,

$$\sup_{0 \le t_0 \le t \le T'} \|S(t, t_0)\| \le C(T')$$

Thus the Duhamel principle implies (24) for all $u \in C^0_{[0,T]}H^s$ solutions of (1), which contradicts the Corollary 25. $\square$

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